Research Article

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Power Topp–Leone exponential negative family of distributions with numerical illustrations to engineering and biological data

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Abstract: This article puts forth a novel category of probability distributions obtained from the Topp–Leone distribution, the inverse-K exponential distribution, and the power functions. To obtain this new family, we used the original cumulative distribution functions. After introducing this new family, we gave the motivations that led us to this end and the basis of the new family obtained, followed by the mathematical properties related to the family. Then, we presented the statistic order, the quantile function, the series expansion, the moments, and the entropy (Shannon, Reiny, and Tsallis), and we estimated the parameters by the maximum likelihood method. Finally, using real data, we presented numerical results through data analysis with a comparison of rival models.

Keywords: inverse-K exponential distribution, power function, entropy, maximum likelihood estimation, statistic order, numerical analysis

1 Introduction

In statistical modeling, the results obtained may not be very close to reality because the chosen model is not adequate for the data. To address this problem, new models have been developed rapidly in order to improve the existing models. Among these models, we can have general families of distributions, most of which are based on generative distributions. Several methods exist for developing novel distributions. Among these methods are the method of generalization of distributions; the method of generation of asymmetric distributions; the method of addition of parameters; the model generated by the beta method; and the method of transformed-transformer. In our work, we have used the method of adding parameters. Indeed, the primary concept revolves around augmenting the fundamental distribution by incorporating one or multiple shape parameters into the basic distribution in order to improve its level of flexibility. We can give the following examples: Poisson-G [1], new power Topp–Leone (TL) generated family of distributions [2], generalized Odd gamma-G family of distributions [3], Topp–Leone–Marshall–Olkin-G family of distributions [4], type II Topp–Leone generated family of distributions [5], Topp–Leone–Gompertz-G family of distributions [6], two-sided generalized Topp and Leone (TS-GTL) distributions [7], Topp–Leone Modified Weibull Model [8], Topp–Leone Lomax (TLLo) distribution [9], Marshall–Olkin–Topp–Leone-G family of distributions [10], type II generalized Topp–Leone family of distribution [11], new power Topp–Leone generated family of distributions [2], logistic-uniform distribution [12], gamma-uniform distribution [13], uniform distribution of Heegner points [14], Topp–Leone odd log-logistic family of distributions [15], type II exponentiated half-logistic-Topp–Leone-G power series class of distributions [16], exponentiated half-logistic-Topp–Leone-G power series class of distributions [17], extended generalized exponential power series distribution [18], new inverted Topp–Leone distribution [19], Kumaraswamy inverted Topp–Leone distribution [20], new power Topp–Leone distribution [21], type II power Topp–Leone Daggum distribution [22], new hyperbolic sine-generator [23], type II Topp–Leone Bur XII distribution [24], exponentiated Topp–Leone distribution [25], Kumaraswamy–Kumaraswamy distribution [26], transmuted Kumaraswamy distribution [27], exponentiated Kumaraswamy distribution [28], inverted Kumaraswamy distribution [29], unit-Weibull distribution as an alternative to the Kumaraswamy distribution [30], inflated Kumaraswamy distributions [31],

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generalized inverted Kumaraswamy distribution [32], bivariate Kumaraswamy distribution [33], Marshall–Olkin extended inverted Kumaraswamy distribution [34], Marshall–Olkin Kumaraswamy distribution [35], Kumaraswamy-geometric distribution [36], Kumaraswamy-log-logistic distribution [37], Kumaraswamy–Pareto distribution [38], Topp–Leone generalized inverted Kumaraswamy distribution [39], two-parameter family of distributions [40], power Lambert uniform distribution [41], and also other families of distributions having important properties and very used in statistical modeling. Moreover, among the existing distributions that are defined on a unit interval, we have the TL distribution, which has great importance in statistics because of its mathematical properties and especially because of the traceability of its cumulative distribution function (CDF). This particular distribution possesses an exceedingly adaptable CDF as it has the potential to represent a positive skewed distribution, a negative skewed distribution, and much more a symmetric distribution, which allows a greater flexibility of the tail: these characteristics distinguish it from others. It can also model hazard rates that are decreasing as well as increasing, bathtub, and inverted J. Another advantage of this distribution of interest is its possession of an exact closed-form CDF, making it highly manageable and straightforward to work with. These very excellent criteria for this distribution make it an elegant candidate for use in various fields. Thus, the *CDF* of TL is defined as follows:

$$F(z_o;\ m) = z_o^m(2 - z_o)^m, \tag{1}$$

where $z_o \in [0, 1]$ and the probability density function (PDF) is characterized as follows:

$$f(z_o;\ m) = 2mz_o^{m-1}(1 - z_o)(2 - z_o)^{m-1} \tag{2}$$

with $z_o \in [0, 1]$.

Therefore, the hazard rate function (hrf) is given as follows:

$$h(z_o;\ m) = \frac{2mz_o^{m-1}\{1 - z_o\}\{2 - z_o\}^{m-1}}{1 - z_o^m\{2 - z_o\}^m}, \tag{3}$$

with $z_o \in [0, 1]$.

This function of the hazard rate is very flexible. This distribution has other mathematical properties that are nontrivial, which makes its application in several fields. Recently, several researchers have used the TL distribution to study the behavior of several aggregates such as taxes and productivity in business. The TL power family is obtained from the TL distribution, the power function, and finally the whole composed by a cumulative distribution. This one is defined as follows:

$$F(x;\ m, n, \xi_o) = K(x;\ \xi_o)^{nm}\{2 - K(x;\ \xi_o)^n\}^m \tag{4}$$

$x \in \mathbb{R};\quad m, n > 0$ with $\xi_o = (\xi_{o1}, \xi_{o2}, \ldots, \xi_{on})$.

Let us introduce the novel family characterized by:

$$F(x;\ \xi) = e^{1 - \frac{1}{K(x;\ \xi_o)^p}}, \tag{5}$$

where $p > 0$.

The idea behind this work is to present a novel family of distributions by merging some of the families described above. It is the power Topp–Leone (PTL-K) and power inverse exponential (PIE-K). The new family obtained is defined by composing their distribution functions. We obtain:

$$F(x;\ \nu) = e^{mn\left\{1 - \frac{1}{K(x;\ \xi_o)^p}\right\}}\left[2 - e^{n\left\{1 - \frac{1}{K(x;\ \xi_o)^p}\right\}}\right]^m, \tag{6}$$

with $\nu = (m, n, p, \xi_o)$.

The novel family thus obtained is: power Topp–Leone exponential negative family of distribution (PTLEN-K).

The development of this family is motivated by several key factors:

- **Inadequacy of existing models:** currently available probability distribution models may prove insufficient for accurately modeling data from the fields of engineering and biology.
- **Specific application:** the identification of specific domains where TL distributions, inverse-K exponential distributions, and power functions are particularly relevant has led to the creation of this new family of distributions.
- **Required flexibility:** the need to offer greater flexibility in modeling real-world data has led to the proposal of these distributions, which is essential for atypical or complex data.
- **Exploration of new theories:** this initiative is part of a research endeavor aimed at expanding knowledge in statistics by exploring new theories and methods.
- **Addressing pending questions:** the resolution of unresolved problems or questions in data modeling in engineering and biology has been a major motivation behind this creation.
- **Practical applications:** the aim of this new family of distributions is to contribute to the improvement of modeling complex phenomena in the fields of engineering and biology, as well as enhance predictions and decisions based on these models.
- **Scientific advancement:** by extending the range of available statistical distributions, this work seeks to promote the advancement of statistical science, benefiting researchers and practitioners working in these fields.

Overall, the creation of the new family of distributions is motivated by the need to provide more suitable statistical tools, address specific problems, and enhance data modeling in specific domains of engineering and biology [42,43].

2 The basics of the PTLEN-K

In this paragraph, we present the fundamental principles of the PTLEN-K family.

2.1 PDF function

The *CDF* function of the PTLEN-K is given as follows:

$$F(x;\,\nu) = e^{mn\left\{1-\frac{1}{K(x;\,\xi_o)^p}\right\}}\left\{2 - e^{n\left\{1-\frac{1}{K(x;\,\xi_o)^p}\right\}}\right\}^m. \tag{7}$$

We differentiate this distribution function according to x. Thus, we obtain:

$$\begin{aligned} f(x;\,\nu) = 2mnp \times \frac{k(x;\,\xi_o)}{K(x;\,\xi_o)^{p+1}} e^{mn\left\{1-\frac{1}{K(x;\,\xi_o)^p}\right\}} \\ \times \left[1 - e^{n\left\{1-\frac{1}{K(x;\,\xi_o)^p}\right\}}\right]\left[\left\{2 - e^{n\left\{1-\frac{1}{K(x;\,\xi_o)^p}\right\}}\right\}\right]^{m-1} \end{aligned} \tag{8}$$

$\forall x \in \mathbb{R}$, where $k(x, \xi_o)$ is the derivative of $K(x;\,\xi_o)$.

Some asymptotic findings on f can be deduced from this expression.

Table 1: Some special members of PTLEN-K

Models	Distribution	$K(x, \xi_o)$	Support
PTLENF	Fréchet	$\exp\{-(-x)^{-\alpha}\}I_{x\geq 0}$	$\mathbb{R}_+$
PTLENG	Gumbel	$\exp\{-e^{-x}\}$	$\mathbb{R}$
PTLENW	Weibull	$1-\exp\{-(-x/\lambda)^{\alpha}\}I_{x\geq 0}$	$\mathbb{R}_+$
PTLE2N	Normal	$\Phi(x;\mu,\sigma)$	$\mathbb{R}$
PTLENLo	Logistique	$\frac{1}{2}\left\{1 + \tanh(\frac{x-\mu}{2s})\right\}$	$\mathbb{R}$
PTLENHC	Half Cauchy	$\frac{2}{\pi}\arctan(\frac{x}{a})$	$\mathbb{R}_+$
PTLENBu	Burr	$1-\{1 + x^c\}^{-k}$	$\mathbb{R}_+$
PTLENLx	Lomax	$1-\left\{1 + \frac{x}{\lambda}\right\}^{-k}$	$\mathbb{R}_+^*$
PTLENBt	Beta	$\frac{B_x(\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$	$(0,1)$
PTLENRay	Rayleigh	$1-\exp\left\{\frac{-x^2}{2a^2}\right\}$	$\mathbb{R}_+$

If $K(x;\,\xi_o) \to 0$:

$$f(x;\,\nu) \sim mnpW_{m,\nu}(x),$$

$$W_{m,\nu}(x) = \frac{k(x;\,\xi_o)}{K(x;\,\xi_o)^{p+1}} \times e^{m\left\{n\left\{1-\frac{1}{K(x;\,\xi_o)^p}\right\}+\log(2)\right\}}.$$

When $K(x;\,\xi_o) \to 1$,

$$f(x;\,\nu) \sim 2mnpZ_\nu(x),$$

with

$$Z_\nu(x) = k(x;\,\xi_o) \times (1 - e^{n(1-\frac{1}{K(x;\,\xi_o)^p})}).$$

2.2 Asymptote of *hrf*

The *hrf* of the PTLEN-K family is:

$$h(x;\,\nu) = \frac{f(x;\,\nu)}{1 - F(x;\,\nu)}, \tag{9}$$

$$\begin{aligned} h(x;\,\nu) = 2 \times mnp \times \frac{k(x;\,\xi_o)}{K(x;\,\xi_o)^{p+1}} \\ \times e^{mn(1-\frac{1}{K(x;\,\xi_o)^p})}\left[1 - e^{n(1-\frac{1}{K(x;\,\xi_o)^p})}\right] \\ \times \frac{\left[2 - e^{n(1-\frac{1}{K(x;\,\xi_o)^p})}\right]^{m-1}}{\left\{1 - e^{mn(1-\frac{1}{K(x;\,\xi_o)^p})}\left[2 - e^{n(1-\frac{1}{K(x;\,\xi_o)^p})}\right]^m\right\}}. \end{aligned}$$

Below are some asymptotic conclusions on h derived from this function.

Proposition 1. *When* $K(x;\,\xi_o) \to 0$,

$$h(x;\,\nu) \sim mnp \times \frac{k(x;\,\xi_o)}{K(x;\,\xi_o)^{p+1}} \times \frac{Y_{m,\nu}(x)}{1 - Y_{m,\nu}(x)} \quad with,$$

$$Y_{m,\nu}(x) = e^{m\left\{n\left\{1-\frac{1}{K(x;\,\xi_o)^p}\right\}+\log(2)\right\}}. \tag{10}$$

Proposition 2. *Moreover, if* $K(x;\,\xi_o) \to 1$,

$$h(x;\,\nu) \sim \frac{2mnp}{1 - F(x)}Z_\nu(x) \quad with$$

$$Z_\nu(x) = k(x;\,\xi_o)\left\{1 - e^{n\left\{1-\frac{1}{K(x;\,\xi_o)^p}\right\}}\right\}.$$

The variations of $h(x;\,\nu)$ can be examined similarly to those of f by utilizing the following relationship:

$$\{\log[h(x;\,\nu)]\}' = \{\log[f(x;\,\nu)]\}' + h(x;\,\nu). \tag{11}$$

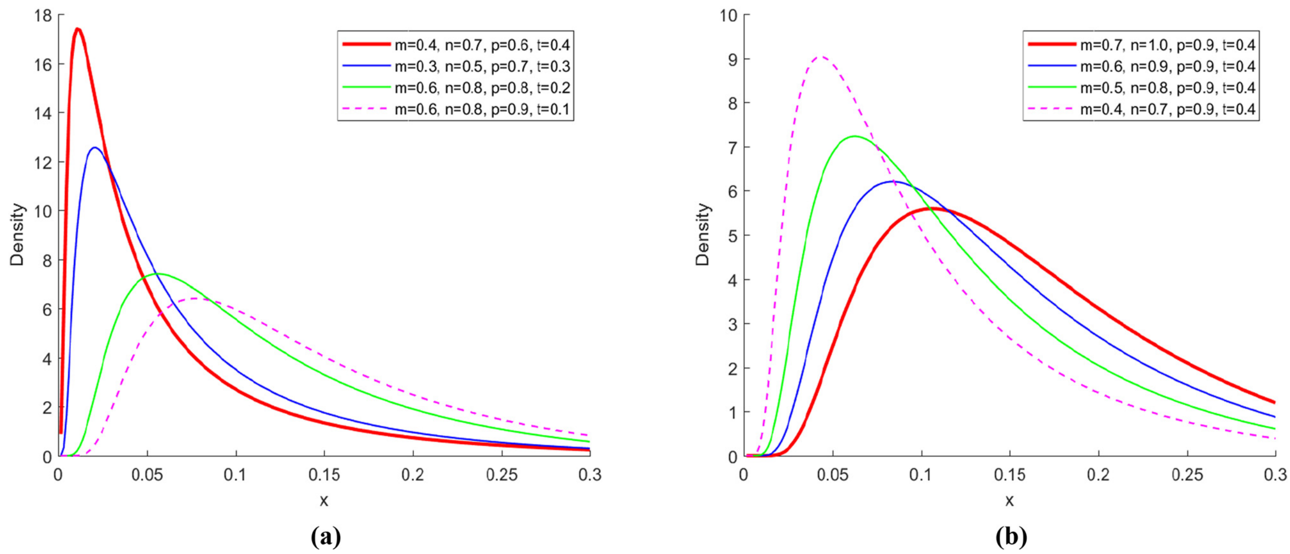


Figure 1: A graphical representation of the empirical *PDFs* for random values of m and n. (a) *PDFs* representation for random values of parameters and (b) *PDFs* representation for fixed p and t and different m and n.

2.3 On a stochastic order: Framing of the proposed new family

The subsequent result demonstrates certain inequalities involving $F(x;\ \nu)$.

Proposition 3. *For all $x \in \mathbb{R}_+^*$ such as $K(x;\ \xi_o) \in (0,1)$, the following inequalities are valid*:

$$G_\nu(x) \le F(x;\ \nu) \le H_\nu(x).$$

Proof. We know that

$$0 \le K(x;\ \xi_o) \le 1.$$

Therefore,

$$mn\left\{1 - \frac{1}{K(x;\ \xi_o)^p}\right\}\left[2 - e^{n\left\{1-\frac{1}{K(x;\ \xi_o)^p}\right\}}\right]^m \le F(x;\ \nu).$$

On the other hand, we know that $e^{n\left\{1-\frac{1}{K(x;\ \xi_o)^p}\right\}} > 0$ because for any $x \in \mathbb{R},\ \ e^x > 0$.

As a result, we have

$$F(x;\ \nu) \le 2^m e^{mn\left\{1-\frac{1}{K(x;\ \xi_o)^p}\right\}}$$

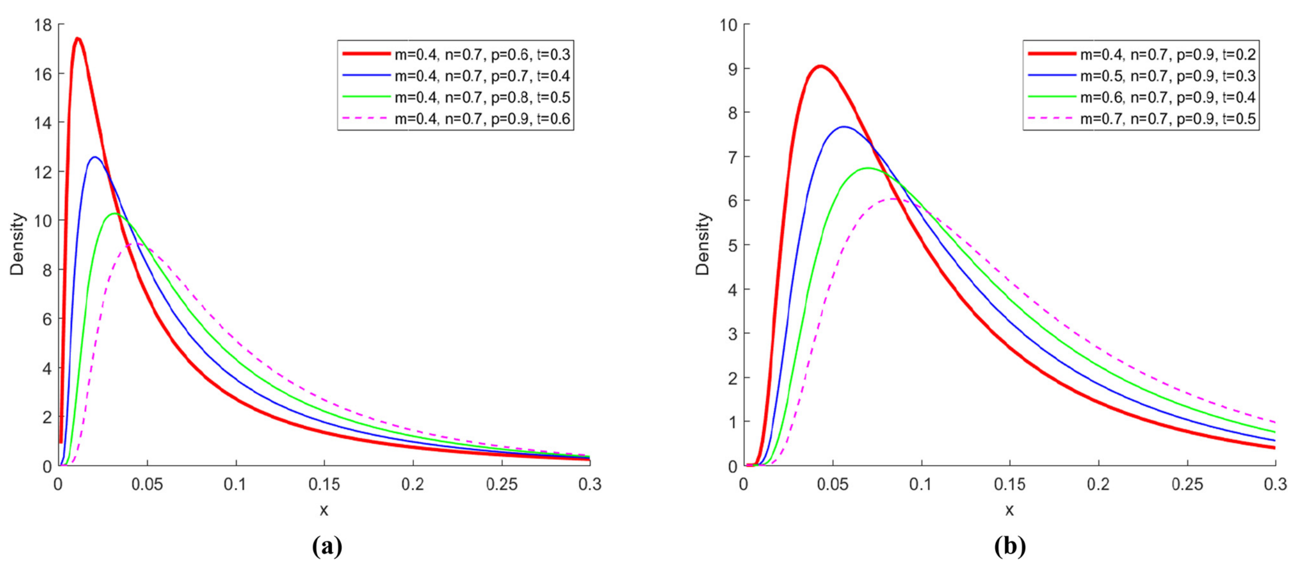
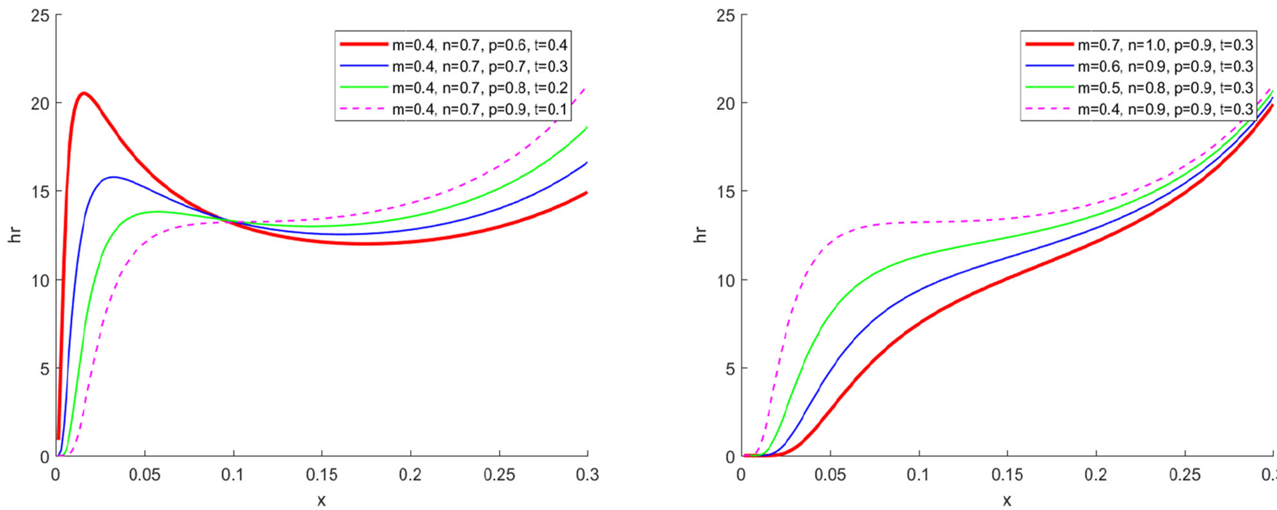


Figure 2: A graphical representation of the empirical *PDFs* for fixed n. (a) *PDFs* representation for fixed m and n and different p and t and (b) *PDFs* representation for fixed p and n and different m and t.



(a) **(b)**

Figure 3: Graphical representation of the empirical *hrfs*. (a) *hrfs* representation for fixed m and n and different p and t and (b) *hrfs* representation for fixed p and t and different m and n.

and

$$G_\nu(x) \le F(x;\ \nu) \le H_\nu(x),$$

with

$$G_\nu(x) = mn\left\{1 - \frac{1}{K(x;\ \xi_o)^p}\right\}\left\{2 - e^{n\left\{1-\frac{1}{K(x;\ \xi_o)^p}\right\}}\right\}^m$$

and

$$H_\nu(x) = 2^m e^{mn\left\{1-\frac{1}{K(x;\ \xi_o)^p}\right\}}.$$

□

3 Special members

From this new distribution family, several are the special members of PTLEN-K with interesting properties. So, we will list in the following table some of these members.

Table 1 shows the special members of PTLEN-K.

4 A special member: the PTLEN-U distribution

Many are the distributions having diverse natures that comprise the new family previously introduced according to the selection of basic probability distribution. In this investigation, we employed a uniform distribution with parameter $\theta > 0$ to establish the characteristics of PTLEN-U. It is determined as follows:

$$K(x;\theta) = \frac{x}{\theta}$$

with $0 < x < \theta$.

The associated PDF and *hrf* are expressed as follows:

$$k(x;\theta) = \frac{1}{\theta}$$

and

$$h(x;\theta) = \frac{1}{\theta - x}.$$

In addition to its simplicity, the PTLEN-U has demonstrated remarkable flexibility in modeling data, exhibiting a nonmonotonic underlying *hrf*. Therefore, the PTLEN-U can be defined by the following CDF:

$$F(x;\ \nu) = e^{mn\left\{1-\left(\frac{\theta}{x}\right)^p\right\}}\left[2 - e^{n\left\{1-\left(\frac{\theta}{x}\right)^p\right\}}\right]^m. \tag{12}$$

The associated PDF and hrf are characterized as follows:

$$f(x;\ \nu) = 2mnp \times \frac{\theta^p}{x^{p+1}} \times e^{mn\left\{1-\left(\frac{\theta}{x}\right)^p\right\}} \times \left\{1 - e^{n\left\{1-\left(\frac{\theta}{x}\right)^p\right\}}\right\}\left\{2 - e^{n\left\{1-\left(\frac{\theta}{x}\right)^p\right\}}\right\}^{m-1} \tag{13}$$

$\forall x \in R$ and $hr(x;\ \nu) = \frac{f(x;\ \nu)}{1 - F(x;\ \nu)}$

$$hr(x;\ v) = 2mnp \times \theta^p e^{mn\left\{1-\left(\frac{\theta}{x}\right)^p\right\}} \times \frac{\left[1 - e^{n\left\{1-\left(\frac{\theta}{x}\right)^p\right\}}\right]\left[2 - e^{n\left\{1-\left(\frac{\theta}{x}\right)^p\right\}}\right]^{m-1}}{x^{p+1}\left\{1 - e^{mn\left\{1-\left(\frac{\theta}{x}\right)^p\right\}}\left[2 - e^{n\left\{1-\left(\frac{\theta}{x}\right)^p\right\}}\right]^m\right\}} \tag{14}$$

with $0 < x < \theta$.

4.1 Some PTLEN-U mathematical properties

In this paragraph, we present several noteworthy mathematical characteristics regarding the distribution of PTLEN-U.

The potential forms of the *PDF* and *hrf* of the PTLEN-U model are illustrated through Figures 1–3. The Figures 1 and 2 display that the PDF may exhibit a right-skewed and inverted J-shaped curve. We also remark that the *PDFs* may take on an increasing, decreasing, inverted, or bathtub-shaped form. These curvature characteristics are widely recognized as advantageous in the development of versatile statistical models.

4.2 Expansion of the function *f*

Proposition 4. *The expansion of f has the following expression*:

$$f(x;\ v) = \frac{2mp}{\theta} \times \sum_{k,j,t,q=0}^{+\infty} \frac{(-1)^{q+t+j} \times n^{1-j} \times 2^{m-1-t}}{k!t!j!} \times \frac{\Gamma(u+1)\Gamma(m)}{\Gamma(u-j+1)\Gamma(m-t)} \times (q+m+t)^k z^{k+j} \tag{15}$$

with $z = n\left\{1 - \left(\frac{\theta}{x}\right)^p\right\}$ *and* $u = \frac{p+1}{p}$

Proof. The *PDF* of PTLEN-U is given in Eq. (13):

Indeed, we are going to pass to the development in series of the factors of this expression:

$$z = n\left\{1 - \left(\frac{\theta}{x}\right)^p\right\} \text{ and } u = \frac{p+1}{p}.$$

Then, the expression of the density function becomes:

$$f(x;\ v) = \frac{2mnp}{\theta}\left(1 - \frac{z}{n}\right)^u e^{mz}(1 - e^z)(2 - e^z)^{m-1}.$$

In addition,

$$(2 - e^z)^{m-1} = \sum_{t=0}^{m-1} (-1)^t \binom{m-1}{t} 2^{m-1-t} e^{zt},$$

So,

$$e^{mz}(2 - e^z)^{m-1} = \sum_{t=0}^{m-1} (-1)^t \binom{m-1}{t} 2^{m-1-t} e^{zt} e^{mz}.$$

In addition, we have

$$(1 - e^z) = \sum_{q=0}^{1} (-1)^q e^{qz}.$$

Consequently,

$$e^{mz}(1 - e^z)(2 - e^z)^{m-1} = \sum_{q,t=0}^{m-1} (-1)^{t+q} \binom{m-1}{t} 2^{m-1-t} e^{(t+m+q)z}$$

$$\text{and} \quad \left(1 - \frac{z}{n}\right)^u = \sum_{j=0}^{u} (-1)^j \binom{u}{j} \left(\frac{z}{n}\right)^j.$$

So,

$$f(x;\ v) = \frac{2mp}{\theta} \sum_{k,j,t,q=0} \times \frac{(-1)^{q+t+j} \times n^{1-j} \times 2^{m-1-t}}{k!t!j!} \times \frac{\Gamma(u+1)\Gamma(m)}{\Gamma(u-j+1)\Gamma(m-t)} \times (q+m+t)^k z^{k+j}.$$

□

4.3 Moments

Proposition 5. *For any random variable X with f as PDF, the moment of X is given by taking S to be positive as follows*:

$$M = \frac{2mp}{\theta} S_{k,j,t,q} M'_{k,s}. \tag{16}$$

Proof.

$$\begin{aligned} M &= E(X^s) \\ &= \int_{-\infty}^{+\infty} x^s f(x;\ v)\mathrm{d}x \\ &= \frac{2mp}{\theta} S_{k,j,t,q} \int_{-\infty}^{+\infty} x^s \left(1 - \left(\frac{\theta}{x}\right)^p\right)^{j+k} \mathrm{d}x \\ M &= \frac{2mp}{\theta} S_{k,j,t,q} M'_{k,s}, \end{aligned}$$

$$S_{k,j,t,q}=\sum_{k,j,t,q=0}\times\frac{(-1)^{q+t+j}\times n^{1-j}\times 2^{m-1-t}}{k!t!j!}\times\frac{\Gamma(u+1)\Gamma(m)}{\Gamma(u-j+1)\Gamma(m-t)}\times(q+m+t)^{k},\tag{17}$$

$$M'_{k,s}=\int_{-\infty}^{+\infty}x^{s}\left(1-\left(\frac{\theta}{x}\right)^{p}\right)^{j+k}\mathrm{d}x.\tag{18}$$

We will, therefore, look for a simpler development of $M'_{k,s}$. Indeed, let us consider $\frac{\theta}{x}=X$ and $\alpha=j+k$

$$\begin{aligned}M'_{k,s}&=\int_{-\infty}^{+\infty}x^{s}\left(1-\left(\frac{\theta}{x}\right)^{p}\right)^{j+k}\mathrm{d}x\\&=\int_{-\infty}^{+\infty}x^{s}(1-X^{p})^{\alpha}\mathrm{d}x\\&=\int_{-\infty}^{+\infty}x^{s}\sum_{d=0}^{\alpha}\binom{\alpha}{d}(-1)^{\alpha-d}X^{p(\alpha-d)}\mathrm{d}x\end{aligned}$$

$$M'_{k,s}=\sum_{d=0}^{\alpha}(-1)^{\alpha-d}\theta^{p(\alpha-d)}\Gamma_{j,d,k}\int_{-\infty}^{+\infty}x^{p(\alpha-d)+s}\mathrm{d}x.\qquad\square$$

4.4 Probabilities weighted moments

Proposition 6. *The* $(r+s)$*th probability weighted moments noted* $M_{r,s}$ *is*:

$$M_{r,s}=\frac{2mp}{\theta}\sum_{k,t,q,j=0}G_{k,t,q,j,m,n,p}M^{o}_{r,\theta,k},\tag{19}$$

with

$$G_{k,t,q,j,m,n,p}=(-1)^{q+t+j}\times\frac{n^{k+1}2^{\gamma-1-t}}{k!t!j!}\times\Gamma_{u,\gamma,j,t}(q+\gamma+t)^{k}$$

$$\Gamma_{u,\gamma,j,t}=\frac{\Gamma(u+1)\Gamma(\gamma)}{\Gamma(u-j+1)\Gamma(\gamma-t)}\tag{20}$$

with $\gamma=m(s+1)$

and

$$M^{o}_{r,\theta,k}=\int_{0}^{+\infty}x^{r}\left(1-\left(\frac{\theta}{x}\right)^{p}\right)^{j+k}\mathrm{d}x.\tag{21}$$

Proof.

$$fF^{s}=\frac{2mnp}{\theta}\left(1-\frac{z}{n}\right)^{u}e^{mz}(1-e^{z})(2-e^{z})^{m-1}[e^{mz}(2-e^{z})^{m}]^{s}$$

$$fF^{s}=\frac{2mnp}{\theta}\left(1-\frac{z}{n}\right)^{u}(1-e^{z})(2-e^{z})^{\gamma-1}e^{\gamma z}.$$

In addition, we have:

$$(2-e^{z})^{m(s+1)-1}=\sum_{t=0}^{m(s+1)-1}(-1)^{t}\binom{\gamma-1}{t}2^{\gamma-1-t}e^{zt}\tag{22}$$

and

$$e^{\gamma z}(2-e^{z})^{\gamma-1}=\sum_{t=0}^{\gamma-1}(-1)^{t}\binom{\gamma-1}{t}2^{\gamma-1-t}e^{zt}e^{\gamma z}.\tag{23}$$

As a result, we have

$$(1-e^{z})=\sum_{q=0}^{1}(-1)^{q}e^{qz}$$

and

$$e^{\gamma z}(1-e^{z})(2-e^{z})^{\gamma-1}=\sum_{q,t=0}(-1)^{t+q}\binom{\gamma-1}{t}2^{\gamma-1-t}e^{(t+\gamma+q)z}.\tag{24}$$

Given,

$$\left(1-\frac{z}{n}\right)^{u}=\sum_{j=0}^{u}(-1)^{j}\binom{u}{j}\left(\frac{z}{n}\right)^{j},$$

we have,

$$fF^{s}=\frac{2mp}{\theta}\times\sum_{k,j,t,q=0}\frac{(-1)^{q+t+j}n^{1-j}2^{\gamma-1-t}}{k!t!j!}\times\frac{\Gamma(u+1)\Gamma(\gamma)(q+\gamma+t)^{k}}{\Gamma(u-j+1)\Gamma(\gamma-t)}z^{j+k}.\tag{25}$$

Consequently,

$$M_{r,s}=E(X^{r}F^{s}(x))=\int_{0}^{+\infty}x^{r}fF^{s}\mathrm{d}x,$$

$$M_{r,s}=\frac{2mp}{\theta}\times\sum_{k,t,q,j=0}\frac{(-1)^{q+t+j}n^{k+1}2^{\gamma-1-t}}{k!t!j!}\times\frac{\Gamma(u+1)\Gamma(\gamma)(q+\gamma+t)^{k}}{\Gamma(u-j+1)\Gamma(\gamma-t)}\times\int_{0}^{+\infty}x^{r}\left(1-\left(\frac{\theta}{x}\right)^{p}\right)^{j+k}\mathrm{d}x.$$

So,

$$M_{r,s} = \frac{2mp}{\theta} \times \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^o_{r,\theta,k}. \tag{26}$$

To simplify $M^o_{r,\theta,k}$, let us consider $\frac{\theta}{x} = X$ and $\alpha = j + k$

$$\begin{aligned} M^o_{r,\theta,k} &= \int_0^{+\infty} x^r (1 - X^p)^\alpha \mathrm{d}x \\ &= \int_0^{+\infty} x^r \sum_{d=0}^{\alpha} \binom{\alpha}{d} (-1)^{\alpha-d} X^{p(\alpha-d)} \mathrm{d}x \\ &= \sum_{d=0}^{\alpha} \binom{\alpha}{d} (-1)^{\alpha-d} \int_0^{+\infty} x^r X^{p(\alpha-d)} \mathrm{d}x \\ M^o_{r,\theta,k} &= \sum_{d=0}^{\alpha} (-1)^{\alpha-d} \theta^{p(\alpha-d)} \Gamma_{j,d,k} \int_0^{+\infty} x^{p(\alpha-d)+r} \mathrm{d}x. \end{aligned}$$

□

4.5 Incomplete moment (IM)

Proposition 7. *X is a random variable and S_o is a positive integer. The S_0-incomplete moment of X is given as follows*:

$$\mathrm{IM}(y) = E(X^{S_o} Y_y)$$

$$\mathrm{IM}(y) = \int_{-\infty}^{y} x^{S_o} f(x;\, v) \mathrm{d}x,$$

where $Y_y = X$ when $X \leq y$ and $Y_y = 0$.

From the expression obtained for the moment given in Eq. (19), we deduce the following IM(y):

$$\mathrm{IM}(y) = \frac{2mp}{\theta} S'_{k,j,t,q} MI'_{k,S_o}, \tag{27}$$

where

$$S'_{k,j,t,q} = \sum_{k,j,t,q=0} \times \frac{(-1)^{q+t+j} \times n^{k+1} \times 2^{m-1-t}}{k!t!j!} \times \Gamma_{u,d,j,t} (q + m + t)^k, \tag{28}$$

with

$$\Gamma_{u,d,j,t} = \frac{\Gamma(u+1)\Gamma(m)}{\Gamma(u-j+1)\Gamma(m-t)} \tag{29}$$

and

$$\mathrm{IM}'_{k,S_o} = \int_{-\infty}^{y} x^{S_o} \left\{1 - \left(\frac{\theta}{x}\right)^p\right\}^{j+k} \mathrm{d}x. \tag{30}$$

On the other hand, we give the reduced form of IM'_{k,S_o}. Let us set $\frac{\theta}{x} = X$ and $\alpha = j + k$. So, we have:

$$\begin{aligned} \mathrm{IM}'_{k,S_o} &= \int_{-\infty}^{y} x^{S_o} \left\{1 - \left(\frac{\theta}{x}\right)^p\right\}^{j+k} \mathrm{d}x \\ &= \int_{-\infty}^{y} x^{S_o} \{1 - X^p\}^\alpha \mathrm{d}x \\ &= \sum_{d=0}^{\alpha} (-1)^{\alpha-d} \theta^{p(\alpha-d)} \Gamma_{j,d,k} \int_{-\infty}^{y} x^{p(\alpha-d)+S_o} \mathrm{d}x \\ \mathrm{IM}'_{k,S_o} &= \sum_{d=0}^{\alpha} (-1)^{\alpha-d} \theta^{p(\alpha-d)} \Gamma_{j,d,k} \int_{-\infty}^{y} x^{p(\alpha-d)+S_o} \mathrm{d}x. \end{aligned}$$

4.6 Moment generating function (MGF)

Proposition 8. *The representation of the MGF can be written as follows*:

$$\mathrm{MGF} = \frac{2mp}{\theta} \sum_{r_o=0}^{\infty} \frac{t^{r_o}}{r_o!} S_{k,j,t,q} M'_{k,S_o}. \tag{31}$$

Proof. We will show the representation of MGF from the series expansion of $\exp(tx)$ as follows:

$$\exp(tx) = \sum_{r_o=0}^{\infty} \frac{(tx)^{r_o}}{r_o!}. \tag{32}$$

Then, $\mathrm{MGF} = E(\exp(tX))$ and

$$\begin{aligned} \mathrm{MGF} &= E(\exp(tX)) \\ &= E\left(\sum_{r_o=0}^{\infty} \frac{(tX)^{r_o}}{r_o!}\right) \\ &= \sum_{r_o=0}^{\infty} \frac{t^{r_o}}{r_o!} E(X^{r_o}) \\ \mathrm{MGF} &= \sum_{r_o=0}^{\infty} \frac{t^{r_o}}{r_o!} M_{r_o} \\ \mathrm{MGF} &= \frac{2mp}{\theta} \sum_{r_o=0}^{\infty} \frac{t^{r_o}}{r_o!} S_{k,j,t,q} M'_{k,S_o}. \end{aligned}$$

□

4.7 Entropies

In statistical modeling, entropy is a measure that studies the variety or vulnerability of a random variable Y. Thus, we have the Rényi entropy, the Shannon entropy, and the Tsallis entropy. We will drive in the following lines: the Shannon's, Renyi's, and Tsallis entropies, which remain the most widely used and have a great importance in many fields, such as statistical inference, classification, problem identification in statistics, econometrics, and pattern recognition in computer science.

4.7.1 Generalized entropy (GE)

Proposition 9. *Cowell and Shorrocks provide that is*:

$$GE(x;\, \beta_o) = \frac{1}{\beta_o(\beta_o - 1)\mu_o^{\beta_o}} \int_0^{\infty} x^{\beta_o} f(x)\mathrm{d}x - 1. \tag{33}$$

With μ_o like the mean of the distribution. So, by using the representation of the probabilities weighted moments, we have:

$$GE(x;\, \beta_o) = \frac{1}{\beta_o(\beta_o - 1)\mu_o^{\beta_o}} G_{k,t,q,j,m,n,p} M^{o}_{r=\beta_o,\theta,k} - 1. \tag{34}$$

4.7.2 Réiny's entropy

For any random and continuous variable X with interval $\mathbb{R}$, the Rényi entropy is a measure of uncertainty. Its definition is as follows:

$$ER(X) = \frac{1}{1-\alpha} \log\left\{\int_R f(x;\, v)^{\alpha}\mathrm{d}x\right\}. \tag{35}$$

Proposition 10. *The Réiny entropy of the PTLEN-U family is given as follows*:

$$ER(X) = \frac{1}{1-\alpha} \log\left\{\sum_{k,t,q,j=0}^{+\infty} J_{k,t,q,j} W_{j,q}(n, \alpha, \theta)\right\}, \tag{36}$$

with

$$J_{k,t,q,j} = \left(\frac{2mnp}{\theta}\right)^{\alpha} \times 2^{\alpha(m-1)-t} \frac{(-1)^{k+t+j}}{k!t!j!q!} \times \Gamma_{u,m,j,k,t} \tag{37}$$

$$\Gamma_{u,m,j,k,t} = \Gamma(au+1) \times \frac{\Gamma(\alpha+1)(k+am+t)^q}{\Gamma(au-j+1)\Gamma(\alpha-k+1)} \times \frac{\Gamma(\alpha(m-1)+1)}{\Gamma(\alpha(m-1)+1-t)}, \tag{38}$$

and

$$W_{j,q}(n, \alpha, \theta) = \int_{\mathbb{R}} n^q \left\{1 - \frac{\theta^p}{x^p}\right\}^{q+j} \mathrm{d}x. \tag{39}$$

Proof. For any random and continuous variable X with interval $\mathbb{R}$, the Rényi entropy is a measure of uncertainty. Its definition is as follows:

$$ER(X) = \frac{1}{1-\alpha} \log\left\{\int_R f(x;\, v)^{\alpha}\mathrm{d}x\right\}.$$

On this, it is essential for us to obtain an explicit expression of $f(x, v)^{\alpha}$. As a matter of fact,

$$f(x;\, v)^{\alpha} = \left(\frac{2mnp}{\theta}\right)^{\alpha} \times 2^{\alpha(m-1)-t} \times \sum_{k,t,q,j=0}^{+\infty} \frac{(-1)^{k+t+j}}{k!t!j!q!} \times \Gamma_{u,j,\alpha} \frac{z^{q+j}}{n^j},$$

where

$$\Gamma_{u,m,j,k,t} = \frac{\Gamma(au+1)\Gamma(\alpha+1)\Gamma(\alpha(m-1)+1)(k+am+t)^q}{\Gamma(au-j+1)\Gamma(\alpha-k+1)\Gamma(\alpha(m-1)+1-t)} \tag{40}$$

and

$$z = n\left\{1 - \left(\frac{\theta}{x}\right)^p\right\}. \tag{41}$$

Consequently, the Rényi entropy of the PTLEN-U distribution is given as follows:

$$ER(X) = \frac{1}{1-\alpha} \log\left\{\sum_{k,t,q,j=0}^{+\infty} J_{k,t,q,j} W_{j,q}(n, \alpha, \theta)\right\}. \tag{42}$$

□

4.7.3 Shanon's entropy

Claude Shannon, a genius researcher, worked at the famous ATT Bell laboratory. He greatly influenced modern science and engineering through his application of thermodynamic techniques to the representation of information. Shannon's genius was to establish the link between the probability of occurrence of a term or character and the "amount of information" associated with it [44].

Proposition 11. *Thus, Shannon entropy of PTLEN-U is defined as follows*:

$$\mathrm{ES} = -\log(m) - \log(n) - \log(p) + \log(\theta) - m\log(2) + \mathrm{ES}_0, \tag{43}$$

$$\mathrm{ES}_0 = -mE(z) + u\sum_{j=1}^{\infty} \frac{1}{jn^j} E(z^j) - (m-1)\sum_{i=1,q=0}^{\infty} \frac{i^{q-1}}{q!2^i} E(z^q) - \sum_{j=1,k=0}^{\infty} \frac{j^{k-1}}{k!} E(z^k).$$

Proof. We know that

$$\mathrm{ES} = -E\{\log[f(X;\, v)]\} \tag{44}$$

$$\mathrm{ES} = -E\{\log(2mnp/\theta)\} - mE(z) - uE\left\{\log(1 - \frac{z}{n})\right\} - E[\log(1 - e^{z})] - (m-1)E\{\log(2 - e^{z})\}$$

and

$$\log(1 - e^{z}) = \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} \frac{j^{k-1}}{k!} z^{k} \tag{45}$$

$$\log(2 - e^{z}) = \log(2) - \sum_{i=1}^{\infty}\sum_{q=0}^{\infty} \frac{i^{q-1}}{q!2^{i}} z^{q}. \tag{46}$$

In addition, we have

$$\mathrm{ES} = -\log(m) - \log(n) - \log(p) + \log(\theta) - m\log(2) + \mathrm{ES}_0, \tag{47}$$

$$\mathrm{ES}_0 = -mE(z) + u\sum_{j=1}^{\infty} \frac{1}{jn^{j}} E(z^{j}) - (m-1)\sum_{i=1,q=0}^{\infty} \frac{i^{q-1}}{q!2^{i}} E(z^{q}) - \sum_{j=1,k=0}^{\infty} \frac{j^{k-1}}{k!} E(z^{k}).$$

□

4.7.4 Tsalli's entropy

Proposition 12. *The entropy of Tsalli's of PTLEN-U is defined as follows*:

$$E_t(\theta) = \frac{1}{1-\lambda}\left\{\sum_{k,t,q,j=0}^{+\infty} J_{k,t,q,j} W_{j,q}(n, \alpha, \lambda)\right\}. \tag{48}$$

Proof.

$$E_t(\theta) = \frac{1}{1-\lambda}\left\{\int f(x)^{\lambda}\mathrm{d}x\right\}. \tag{49}$$

So, by using the representation of the probability weighted moment given in Eq. (19), we have Eq. (48). □

4.8 Quantile function (Q)

Proposition 13. *The Q of PTLEN-K is expressed as follows*:

$$Q(y;\, \nu) = \frac{\theta}{\left[1 - \frac{1}{n}\ln\left(1 - \sqrt{1 - y^{\frac{1}{m}}}\right)\right]^{\frac{1}{p}}}. \tag{50}$$

Proof. Indeed, let us say $x_y = Q(y;\, \nu)$ $\forall y \in [0, 1]$.

Then, by the definition of the quantile function, x_y satisfies the nonlinear equation $y = F(x;\, \nu)$

$$y = F(x;\, \nu) \tag{51}$$

$$y^{\frac{1}{m}} = e^{n(1-\frac{1}{K(x;\, \xi_o)^{p}})}\left[2 - e^{n(1-\frac{1}{K(x;\, \xi_o)^{p}})}\right]. \tag{52}$$

Let us take $w = e^{n\left\{1-\frac{1}{K(x;\, \xi_o)^{p}}\right\}}, y > 0.$

The equation amounts to

$$y^{\frac{1}{m}} = w(2 - w) = 2w - w^{2} \tag{53}$$

and

$$w^{2} - 2w + y^{\frac{1}{m}} = 0. \tag{54}$$

This equation has the following solutions:

$w_1 = 1 + \sqrt{1 - y^{\frac{1}{m}}}$ and $w_2 = 1 - \sqrt{1 - y^{\frac{1}{m}}}$.

The antecedent of $[0, 1]$ is $w \mapsto (2 - w)w$ in $[0, 1]$. So,

$$K(x;\, \xi) = \frac{1}{\left[1 - \frac{1}{n}\ln\left(1 - \sqrt{1 - y^{\frac{1}{m}}}\right)\right]^{\frac{1}{p}}}. \tag{55}$$

But in our case,

$$K(x;\theta) = \frac{x}{\theta}. \tag{56}$$

So,

$$\frac{x}{\theta} = \frac{1}{\left[1 - \frac{1}{n}\ln\left(1 - \sqrt{1 - y^{\frac{1}{m}}}\right)\right]^{\frac{1}{p}}}. \tag{57}$$

By drawing x in this last expression, we have:

$$x = \frac{\theta}{\left[1 - \frac{1}{n}\ln\left(1 - \sqrt{1 - y^{\frac{1}{m}}}\right)\right]^{\frac{1}{p}}}. \tag{58}$$

Hence, the quantile function is given as follows:

$$Q(y;\, \nu) = \frac{\theta}{\left[1 - \frac{1}{n}\ln\left(1 - \sqrt{1 - y^{\frac{1}{m}}}\right)\right]^{\frac{1}{p}}}. \tag{59}$$

□

4.9 Reliability properties

This paragraph covers fundamental reliability properties of the PTLEN-U model that are commonly utilized in probability theory and engineering.

4.9.1 Reliability function or survival function

The reliability function (survival function) is defined as follows:

$$S = 1 - F(x) \tag{60}$$

$$S_{\text{PTLEN-U}} = 1 - e^{mz}[2 - e^{z}]^{m}. \tag{61}$$

4.9.2 Hazard function

The HF function is defined as follows:

$$\text{HF} = \frac{f(x)}{S(x)}, \tag{62}$$

$$\text{HF}_{\text{PTLEN-U}} = \frac{2mnp}{\theta} \times \left(1 - \frac{z}{n}\right)^{u} e^{mz} \times \frac{(1 - e^{z})(2 - e^{z})^{m-1}}{1 - e^{mz}(2 - e^{z})^{m}}.$$

4.9.3 Cumulative hazard function

The cumulative HF is defined as follows:

$$\text{HF}(x) = -\log(S(x)). \tag{63}$$

So, we have

$$\text{HF}_{\text{PTLEN-U}}(x) = -\log[1 - e^{mz}(2 - e^{z})^{m}]. \tag{64}$$

4.9.4 Reserve hazard function

The reserve hazard function is defined as follows:

$$RF(x) = \frac{f(x)}{F(x)}. \tag{65}$$

So, we have

$$RF_{\text{PTLEN-U}} = \frac{2mnp}{\theta}\left(1 - \frac{z}{n}\right)^{u} \frac{1 - e^{z}}{2 - e^{z}}. \tag{66}$$

4.9.5 Mean waiting time (MWT)

Proposition 14. *The MWT function is defined as follows*:

$$\pi(x) = x - \left\{\frac{1}{F(x)}\int_{0}^{x} tf(t)\mathrm{d}t\right\}. \tag{67}$$

So, by using the representation of the probability weighted moments given in Eq. (19), *we have*:

$$\pi(x) = x - \frac{2mp}{\theta}\frac{1}{F(x)}\sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p}M^{o}_{r=1,\theta,k}(x). \tag{68}$$

4.9.6 Mean residual life

Proposition 15. *The mean residual life function (MRL) is defined as follows*:

$$\text{MRL}(x) = \frac{1}{S(x)}\int_{x}^{\infty} tf(t)\mathrm{d}t - x. \tag{69}$$

So, by using the representation of the probability weighted moments given in Eq. (19), *we have*

$$\text{MRL}(x) = \frac{2mp}{\theta} \times \frac{1}{1 - F(x)} \times \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p}M^{o}_{r=1,\theta,k}(x) - x. \tag{70}$$

4.10 Income inequality measures

The uniform model finds practical application in several areas. Consequently, it is crucial to examine some inequality measures that are commonly used in this domain. These measures are also employed in demographic research, which enhances the versatility of the PTLEN-U distribution and expands its scope of application. The inequality measures for this purpose are presented in the following subsections.

4.10.1 Gini index

Italian statiscian Corrado Gini (1912) introduced the following inequality:

$$G_{id} = \frac{1}{\mu_{o}}\int_{0}^{\infty}\{F(x)(1 - F(x))\}\mathrm{d}x, \tag{71}$$

where μ_o and $F(x)$ are the mean CDF, respectively.

Proposition 16. *So, the Gini index of PTLEN-U is given as follows*:

$$G_{id} = B_1\int_{0}^{\infty} z^{q}\mathrm{d}x + B_2\int_{0}^{\infty} z^{d}\mathrm{d}x.$$

Proof. We have:

$$F(x)^2 = \sum_{k,d=0} (-1)^k \binom{2m}{k} 2^{2m-k} \frac{(2m+k)^d}{d!} z^d. \quad (72)$$

So,

$$G_{id} = \frac{1}{\mu_o} \int_0^\infty \{F(x)(1 - F(x))\} \mathrm{d}x$$
$$= \frac{1}{\mu_o} \int_0^\infty F(x) \mathrm{d}x - \frac{1}{\mu_o} \int_0^\infty F(x)^2 \mathrm{d}x$$
$$G_{id} = B_1 \int_0^\infty z^q \mathrm{d}x + B_2 \int_0^\infty z^d \mathrm{d}x$$

with

$$B_1 = \sum_{t,q=0} \times (-1)^t \binom{m}{t} 2^{m-t} \frac{(m+t)^q}{\mu_o q!}$$
$$B_2 = \sum_{k,d=0} (-1)^k \binom{2m}{k} 2^{2m-k} \frac{(2m+k)^d}{\mu_o d!}. \qquad \square$$

4.10.2 Lorenz curve

Proposition 17. *Lorenz introduced another inequality measure as follows*:

$$L(x) = \frac{1}{\mu_o} \int_0^x t f(t) \mathrm{d}t. \quad (73)$$

So, by using the representation of the probability weighted moments given in Eq. (19), *we have*:

$$L(x) = \frac{1}{\mu_o} \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^o_{r=1,\theta,k}(x). \quad (74)$$

4.10.3 Bonferroni index

Proposition 18. *The credit for introducing this inequality goes to Bonferroni. It is derived as the quotient of the Lorenz curve and the CDF.*

$$B(x) = \frac{L(x)}{F(x)}. \quad (75)$$

So, by using the representation of the probability weighted moments given in Eq. (19), *we have*:

$$B(x) = \frac{1}{\mu_o F(x)} \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^o_{r=1,\theta,k}(x). \quad (76)$$

4.10.4 Average deviation from mean

The mean deviation, also known as the average deviation, is calculated as the average of the absolute differences between each value and the mean of those values.

Proposition 19. *The average deviation from mean of PTLEN-U is given as follows*:

$$\psi(\mu_o) = 2\left\{\mu_o F(\mu_o) - \frac{2mp}{\theta} \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^o_{r=1,\theta,k,\mu_o}\right\}. \quad (77)$$

Proof.

$$\psi(\mu_o) = 2\left\{\mu_o F(\mu_o) - \int_0^{\mu_o} x f(x) \mathrm{d}x\right\}. \quad (78)$$

Thus,

$$\psi(\mu_o) = \int_0^\infty |x - \mu_o| f(x) \mathrm{d}x$$
$$= \int_0^{\mu_o} |x - \mu_o| f(x) \mathrm{d}x + \int_{\mu_o}^\infty |x - \mu_o| f(x) \mathrm{d}x$$
$$= 2\mu_o F(\mu_o) - 2\int_0^{\mu_o} x f(x) \mathrm{d}x$$
$$\psi(\mu_o) = 2\left\{\mu_o F(\mu_o) - \int_0^{\mu_o} x f(x) \mathrm{d}x\right\}.$$

By using the representation of the probability weighted moments given in Eq. (19), we have:

$$\int_0^{\mu_o} x f(x) \mathrm{d}x = \frac{2mp}{\theta} \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^o_{r=1,\theta,k,\mu_o}. \quad (79)$$

Consequently,

$$\psi(\mu_o) = 2\left\{\mu_o F(\mu_o) - \frac{2mp}{\theta} \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^o_{r=1,\theta,k,\mu_o}\right\}. \square \quad (80)$$

4.10.5 Pietra index

Proposition 20. *This index was introduced by Pietra. It is defined as the ratio of the mean deviation from the mean to twice the mean of the distribution.*

$$\mathrm{PI} = \frac{\psi(\mu_o)}{2\mu_o}. \quad (81)$$

So, by using the representation given in Eq. (19), *the Pietra index of PTLEN-U is*:

$$\mathrm{PI} = F(\mu_o) - \frac{2mp}{\theta\mu_o}\sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^{o}_{r=1,\theta,k,\mu_o}. \tag{82}$$

4.10.6 Zenga index

Proposition 21. *The following inequality measure called Zenga index was given by Zenga as follows*:

$$Z(x) = 1 - \frac{\mu_o(x)^-}{\mu_o(x)^+}, \tag{83}$$

where

$$\mu_o(x)^- = \frac{1}{F(x)}\int_0^x tf(t)\mathrm{d}t, \tag{84}$$

and

$$\mu_o(x)^+ = \frac{1}{1-F(x)}\int_x^\infty tf(t)\mathrm{d}t, \tag{85}$$

$$\mu_o(x)^- = \frac{2mp}{\theta} \times \frac{1}{F(x)} \times \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^{o}_{r=1,\theta,k}(x^-), \tag{86}$$

$$\mu_o(x)^+ = \frac{2mp}{\theta} \times \frac{1}{1-F(x)} \times \sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^{o}_{r=1,\theta,k}(x^+). \tag{87}$$

So, we have

$$Z(x) = 1 - \frac{1-F(x)}{F(x)} \times \frac{\sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^{o}_{r=1,\theta,k}(x^-)}{\sum_{k,t,q,j=0} G_{k,t,q,j,m,n,p} M^{o}_{r=1,\theta,k}(x^+)}. \tag{88}$$

4.11 Maximum likelihood estimation (MLE)

In this part, we look at the PTLEN-U model. The MLE technique is employed to obtain m, n, p, and t values because of its intriguing theoretical and practical qualities.

The likelihood is given as follows:

$$L_o(\nu) = \prod_{k=1}^{N} f(x_k;\, \nu) \tag{89}$$

$$l_o(\nu) = \log[L_o(\nu)]. \tag{90}$$

Therefore, the log-likelihood is:

$$\begin{aligned} l_o(\nu) &= N\log(2) + N\log(n) + N\log(m) + N\log(p) \\ &\quad + Np\log(\theta) \\ &\quad + mnN - (p+1)\sum_{k=1}^{N}\log(x_k) - mn\theta^p\sum_{k=1}^{N}\frac{1}{x_k^p} \\ &\quad + \sum_{k=1}^{N}\log(1-e^{z_k}) + (m-1)\sum_{k=1}^{N}\log(2-e^{z_k}). \end{aligned} \tag{91}$$

So, the MLEs are defined as follows:

$$\frac{\partial l_o}{\partial m} = \frac{N}{m} + nN - n\theta^p\sum_{k=1}^{N}\frac{1}{x_k^p} + \sum_{k=1}^{N}\log(2-e^{z_k}), \tag{92}$$

$$\begin{aligned} \frac{\partial l_o}{\partial n} &= \frac{N}{n} + mN - m\theta^p\sum_{k=1}^{N}\frac{1}{x_k^p} - \sum_{k=1}^{N}\frac{\left(1-\left(\frac{\theta}{x_k}\right)^p\right)e^{z_k}}{1-e^{z_k}} \\ &\quad - (m-1)\sum_{k=1}^{N}\frac{\left(1-\left(\frac{\theta}{x_k}\right)^p\right)e^{z_k}}{2-e^{z_k}}, \end{aligned} \tag{93}$$

$$\begin{aligned} \frac{\partial l_o}{\partial p} &= \frac{N}{p} + N\log(\theta) - \sum_{k=1}^{N}\log(x_k) \\ &\quad - mn\theta^p\sum_{k=1}^{N}\frac{1}{x_k^p}\log\left(\frac{\theta}{x}\right) \\ &\quad + n\theta^p\sum_{k=1}^{N}\frac{e^{z_k}\log\left(\frac{\theta}{x_k}\right)}{x_k^p\{1-e^{z_k}\}} \\ &\quad + (m-1)n\theta^p\sum_{k=1}^{N}\frac{\log\left(\frac{\theta}{x_k}\right)e^{z_k}}{x_k^p\{2-e^{z_k}\}}, \end{aligned} \tag{94}$$

$$\begin{aligned} \frac{\partial l_o}{\partial \theta} &= \frac{Np}{\theta} - mnp\theta^{p-1}\sum_{k=1}^{N}\frac{1}{x_k^p} + np\theta^{p-1}\sum_{k=1}^{N}\frac{e^{z_k}}{x_k^p\{1-e^{z_k}\}} \\ &\quad + (m-1)np\theta^{p-1}\sum_{k=1}^{N}\frac{e^{z_k}}{x_k^p\{2-e^{z_k}\}}. \end{aligned} \tag{95}$$

These expressions are complex and do not enable us to obtain really closed forms for the MLEs. We will thus make use of numerical methods to maximize $l_o(\nu)$ based on Newton–Raphson algorithms.

5 Simulation with real data

Here, we study the flexibility of PTLEN-U model through an examination of two datasets obtained from real-life incidents. Furthermore, we conducted a comparison between the PTLEN-U model and several other models, a few of

Table 2: Estimated values for the dataset I

Model	m	n	p	θ	a	b	η
PTLEN-U	0.00149	2.4359	5.690	3.700	—	—	—
TL-OEHL-U	—	—	—	—	0.0024	3.8853	2.235
TIIGTLU	—	—	—	19.895	1.425	1.501	—
Uniform	—	—	—	—	0.0011	1.6011	—

which are enumerated below, in order to evaluate its suitability for the data.

(1) Topp–Leone odd half-logistic uniform (TL-OEHL-U) [45].
(2) Type II generalized Topp–Leone-uniform (TIIGTLU) [11].
(3) Uniform (U).

The fundamental similarity among these distributions is their usage of the uniform distribution as a base, thereby enabling a comparison between these models. Several widely recognized statistical metrics and various others, were employed to contrast and evaluate these distributions. It is worth noting that the model with the lowest criterion is considered the most optimal. To calculate all these metrics, we utilized both MATLAB and Mathematica software.

A variety of metrics are employed to evaluate and contrast the four proposed models. The designated criteria consist of akaike information criterion (AIC), Bayesian information criterion (BIC) corrected akaike information criterion (CAIC), Hannan–Quinn information criterion (HQIC), and L_o, which are defined as follows:

$$\text{AIC} = 2b - 2\log(L_o),$$

$$\text{BIC} = b\log(n_o) - 2\log(L_o),$$

$$\text{CAIC} = \text{AIC} + \frac{2b(b+1)}{n_o - b - 1},$$

$$\text{HQIC} = 2b\log[\log(n_o)] - 2\log(L_o),$$

where b is the number of parameters in the statistical model, n_0 denotes the sample size, and L_o signifies the maximized value of the log-likelihood function under the considered model.

Dataset I

The first dataset consists of 63 observations of the strengths of 1.5 cm glass fibers obtained by workers at the UK National Physical Laboratory in [46]. The data are: 1.250, 1.270, 1.280, 1.290, 1.300, 1.360, 1.390, 1.420, 1.480, 1.480, 1.490, 1.490, 1.500, 1.500, 1.510, 1.520, 1.530, 1.540, 1.550, 1.550, 1.580, 1.590, 1.600, 1.610, 1.610, 1.620, 1.630, 1.640, 1.660, 1.660, 1.670, 1.680, 1.690, 1.700, 1.730, 1.760, 1.770, 1.780, 1.810, 1.820, 1.840, 1.840, 1.890, 2.000, 2.010, and 2.240.

Tables 2 and 3, show, respectively, the estimates of the parameters and the criteria for dataset I.

Dataset II

The second dataset represents the survival time (in days) of some guinea pigs infected with virulent tubercle bacilli, observed and reported by as given in the study by Soliman *et al.* [47]:

0.100, 0.330, 0.440, 0.560, 0.590, 0.720, 0.740, 0.770, 0.920, 0.930, 0.960, 1.000, 1.000, 1.020, 1.050, 1.070, 1.070, 1.080, 1.080, 1.080, 1.090, 1.120, 1.130, 1.150, 1.160, 1.200, 1.210, 1.220, 1.220, 1.240, 1.300, 1.340, 1.360,1.390, 1.440, 1.460, 1.530, 1.590, 1.600, 1.630, 1.630, 1.680, 1.710, 1.720, 1.760, 1.830, 1.950, 1.960, 1.970, 2.020, 2.130, 2.150, 2.160, 2.220, 2.300, 2.310, 2.400, 2.450, 2.510, 2.530, 2.540, 2.540, 2.780, 2.930, 3.270, 3.420, 3.470, 3.610, 4.020, 4.320, 4.580, and 5.550.

Tables 4 and 5 show, respectively, the estimators of the parameters and the criteria for dataset II.

Table 4: Estimated values for the dataset II

Model	m	n	p	θ	a	b	η
PTLEN-U	0.024	2.235	1.922	5.885	—	—	—
TL-OEHL-U	—	—	—	—	2.952	10.015	7.405
TIIGTLU	—	—	—	2.012	1.204	1.015	—
Uniform	—	—	—	—	2.314	12.027	—

Table 3: The information criteria results for the hailing time data

Models	$\hat{l}$	AIC	CAIC	BIC	HQIC
PTLEN-U	5.282859	18.56572	19.54133	25.88028	21.3058
TL-OEHL-U	9.81204	25.62408	26.19551	31.1100	27.67914
TIIGTLU	8.98752	23.97504	24.54647	29.46096	26.0301
Uniform	21.86521	45.73042	45.82133	47.55906	46.41544

Table 5: The information criteria results for data II

Model	$\hat{l}$	AIC	CAIC	BIC	HQIC
PTLEN-U	226.0052	460.0105	460.6074	469.1171	463.6359
TL-OEHL-U	242.8501	491.7002	492.0531	498.5302	494.4192
TIIGTLU	253.8640	513.728	514.0809	520.558	516.447
Uniform	310.5014	623.0028	623.0599	625.2795	623.9091

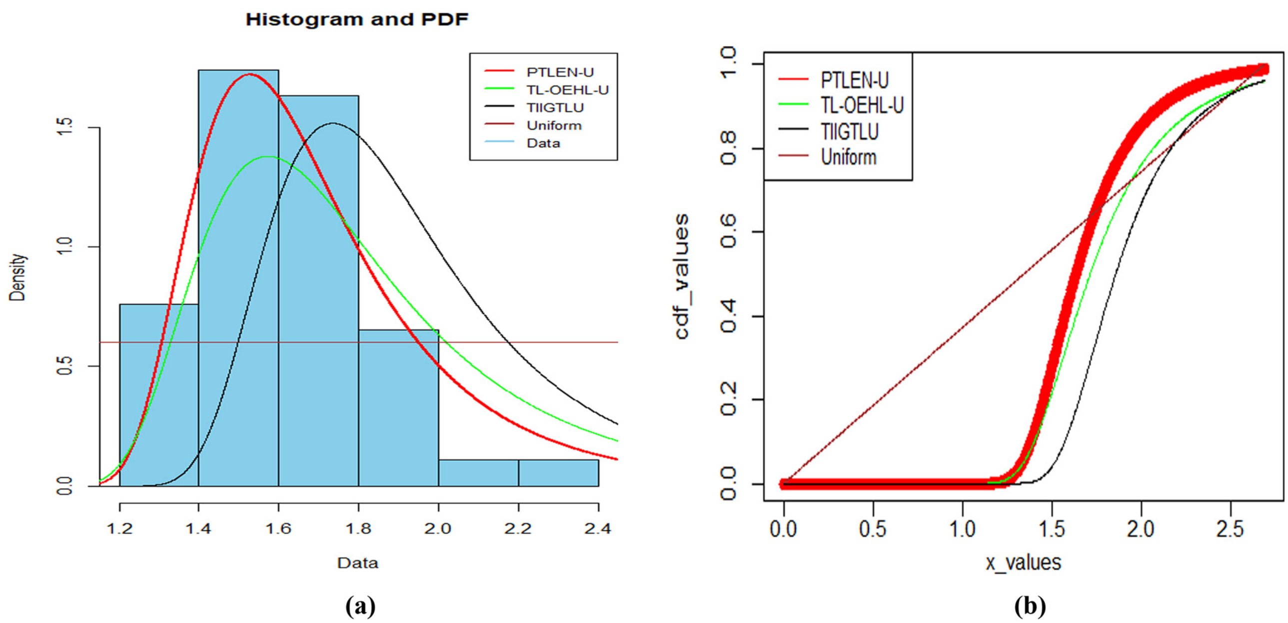


Figure 4: PDFs and CDFs (dataset I). (a) PDFs and (b) CDFs.

Based on the analysis of Figures 4 and 5, we can infer that the PTLEN-U model is a better fit for datasets I and II compared to the TIIGTLU, TL-OEHL-U, and uniform models. One advantage of the PTLEN-U model is its flexibility.

Therefore, we can conclude that the PTLEN-U model is a more suitable choice for modeling these datasets due to its superior performance and versatility in accommodating different data. It can adapt to complex and heterogeneous distributions, which is essential in fields such as engineering and biology where data can exhibit diverse characteristics. By offering better data fits to real data, our method can contribute to more informed decision-making in fields where critical decisions are made based on statistical models. This can have a positive impact on decision quality and forecasting.

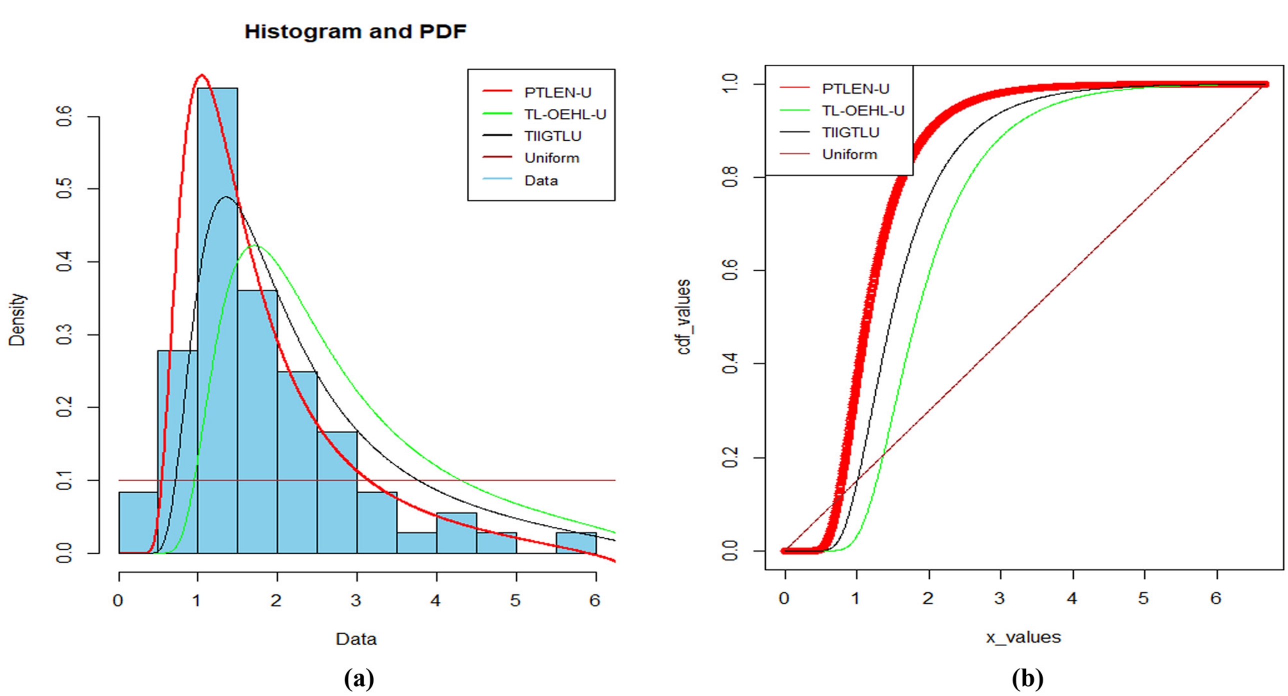


Figure 5: PDFs and CDFs (dataset II). (a) PDFs and (b) CDFs.

6 Conclusion

In this study, we introduced and analyzed a novel distribution known as the PTLEN-U. The PTLEN-U model is derived by incorporating the inverse uniform distribution into the PTL-K family, making it a new contribution to the theory of statistical distributions. The versatility of this distribution allows it to be adapted for various fields of application.

We have explored in our study many mathematical properties of the PTLEN-U model such as Rényi entropy, Tsallis entropy, *qf*, reliability properties, and moment generating functions.

MLE method was employed to obtain the values of unknown parameters of the PTLEN-U model. Furthermore, we applied the PTLEN-U model to two practical datasets and compared its performance to that of its competitors.

Overall, we believe that the PTLEN-U distribution is able to be highly useful for a wide range of real-world data beyond the scope of this study. The creation of new family models would allow for the development of more advanced statistical models, which could potentially lead to improved analysis and predictions in various fields of application.

7 Future work and upcoming studies

In the future, our research team plans to focus on several areas of investigation. One of these areas will involve exploring the $T - X$ transformation to develop a novel distribution able to model previously unobserved lifetime events. This new distribution will offer a more advanced statistical model for analyzing and predicting lifetimes, and it has the potential to be used in a variety of fields.

Another area of investigation will be the development of a bivariate distribution, which will enable us to analyze and predict the joint behavior of two variables. We will also study copulas and other properties of this new distribution, which will allow us to better understand its behavior and applications.

Finally, we plan to apply the novel model to medicine data. This will provide insights into how the new model can be applied in real-world scenarios and will help to further establish its potential usefulness in industry and other fields.

Overall, our future research will focus on developing advanced statistical models and analyzing their behavior in various applications. We look forward to the potential insights and advancements that may result from these investigations.

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